Math 206A Lecture 7 Notes

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1 Borsuk's Conjecture and the Kahn-Kalai Theorem

1.1 Borsuk's conjecture

Here is Borsuk's conjecture.

Theorem 1.1 (Borsuk). *For all convex $X \subseteq \mathbb{R}^d$, there exists a decomposition $X = \bigcup_{i=1}^{d+1} X_i$ such that $\operatorname{diam}(X_i) < \operatorname{diam}(X)$.*

Borsuk showed that this holds for $d = 2$, and it was later shown that this holds in $d = 3$. However, the conjecture is false.

Theorem 1.2 (Kahn-Kalai,1993). *For all $d > 2000$, there exists $X \subseteq \mathbb{R}^d$ such that for all $X = \bigcup_{i=1}^{N} X_i$, $\operatorname{diam}(X_i) < \operatorname{diam}(X) \implies N > c^{\sqrt{d}}$ for some $c > 1$.*

We will prove this. First, let us prove a theorem.

Theorem 1.3 (Pál). *Let X be the unit ball. Then the minimum number of compact sets in the decomposition is $d + 1$.*

Proof. We have already shown that $N \leq d + 1$. We need to show that $N > d$. Look at proposition 3.4 in the textbook. The general proof uses the Borsuk-Ulam theorem from topology. □

1.2 Proof of the Kahn-Kalai theorem

Let's now prove the Kahn-Kalai theorem, which refutes Borsuk's conjecture in general. There have a sequence of simplifications by K-K, Alon[1], Aigner-Zieglar, then Skopenkov. We will see the Skopenkov version of the proof.

[1] Alon published on a pseudoynm: Nilli, the name of his daughter.

Proof. Let $M = \{(x_1, \dots, x_n) \in \mathbb{R}^N : x_i \in \{\pm 1\}, x_1 = 1, x_2 \cdots x_n = 1\}$. Then $|M| = 2^{n-2}$. Let $f : M \to \mathbb{R}^{n^2}$ be $F(x_1, \dots, x_n) = (x_i \cdots x_j)_{1 \leq i,j \leq n}$. So we take a vector and get a matrix. For example,

$$F(1, -1, -1) = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

The construction is $F(M) \to X$. The idea is we can't separate these 2^{n-2} points in X. We need a few lemmas. □

Lemma 1.1. *For $x_i, y_i \in M$,*

$$(x_i x_j - y_i y_j)^2 = (1 - x_i x_j y_i y_j)^2$$

Proof.

$$(x_i x_j - y_i y_j)^2 = (x_i x_j)^2 (1 - x_i^{-1} x_j^{-1} y_i y_j)^2 = (1 - x_i x_j y_i y_j)^2.$$

□

Let's continue with our proof of the Kahn-Kalai theorem.

Proof. Let $n - a$ be the Hamming distance $(\overline{x}, \overline{y})$ i.e. a is the number of i such that $x_i = y_i$. This is the number of $x_i y_i$ that equal 1. So

$$\begin{aligned} d(f(\overline{x}) f(\overline{y}))^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i x_j - y_i y_j)^2 \\ &= \sum_{i} \sum_{j} (1 - x_i y_i x_j y_j) \\ &= 8a(n - a) \end{aligned}$$

This is maximized at $a = n/2$, which is equivalent to $\overline{xy} = 0$. We will continue this next time. □